Nonlinear Finite Element Analysis Including Higher-Order Strain Energy Terms

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In a recent paper,[1] the authors of the present Note presented a formulation for the geometrically nonlinear analysis of shells of revolution by the matrix displacement method. It was assumed that for the majority of problems, certain higher-order nonlinearities in the strain energy expression could be neglected. However, subsequent investigation has led to the conclusion that these second-order effects may be quite important. It is the purpose of this Note to present numerical examples with these higher-order terms included.

Following the notation of Ref. 1, the strain energy of the shell may be separated into two parts:

$$U = U_L + U_{NL} \tag{1}$$

The first term U_L is the usual expression for the strain energy based on linear theory. The second term U_{NL} is due to the nonlinearities in the strain-displacement relations. Assuming that only those nonlinearities due to rotations about the shell coordinate axes are important, U_{NL} may be written:

$$U_{NL} = \tfrac{1}{2}\iint \{C_1\hat{e}_s\hat{e}_{13}^2 + C_2\hat{e}_\theta\hat{e}_{23}^2 + \nu_{s\theta}C_1(\hat{e}_s\hat{e}_{23}^2 + \hat{e}_\theta\hat{e}_{13}^2) + 2G_1\hat{e}_{s\theta}\hat{e}_{13}\hat{e}_{23} + [\tfrac{1}{4}C_1\hat{e}_{13}^4 + \tfrac{1}{4}C_2\hat{e}_{23}^4 + (\tfrac{1}{2}\nu_{s\theta}C_1 + G_1)\hat{e}_{13}^2\hat{e}_{23}^2]\}rdsd\theta \tag{2}$$

where $\hat{e}_s$, $\hat{e}_\theta$, and $\hat{e}_{s\theta}$ are the midsurface meridional, circumferential, and shear strain expressions based on linear theory; $\hat{e}_{13}$ and $\hat{e}_{23}$ are the rotations about the circumferential and meridional coordinate axes, respectively; C_1, C_2, G_1, and $\nu_{s\theta}$ are material constants; and s and θ are meridional and circumferential coordinates.

In Ref. 1, the rotations raised to the fourth power in the square brackets of Eq. (2) were neglected in the strain energy expression [Eq. (13) of Ref. 1]. These fourth-order terms were initially neglected for several reasons: 1) similar terms were neglected in previous beam, plate, and shell finite element analyses on the assumption that they were small in comparison to the strains cubed, 2) a considerable amount of computer execution time is saved, and 3) programing is considerably simplified. However, following publication of Ref. 1, it has been found that omission of the fourth-order terms yields quite conservative results even for those problems which were thought to be only moderately nonlinear. For highly nonlinear problems near the buckling load, the omission of the fourth-order terms may result in completely erroneous results.

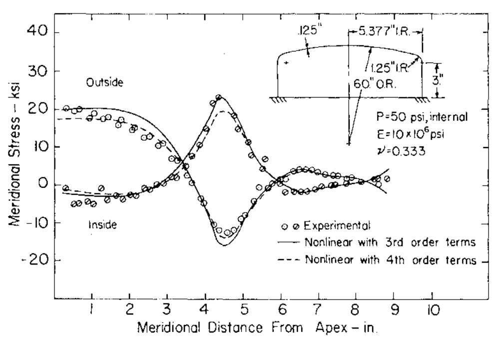

Fig. 1 Meridional stress along arc length.

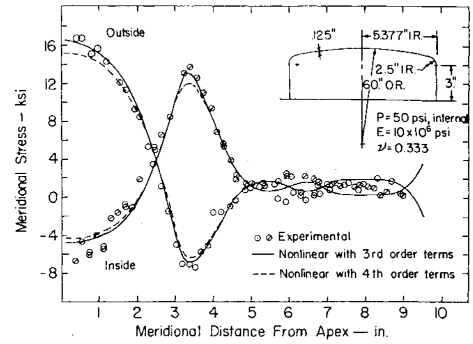

Fig. 2 Meridional stress along arc length.

The rotations raised to the fourth power have been incorporated into the SNASOR[2] computer code. The following results are for the same problems presented in Ref. 1 and clearly illustrate how important the higher-order terms may be.

Figures 1 and 2 present a comparison of experimental and nonlinear meridional stress for two test specimens under internal pressure loading (see Ref. 1 for detailed geometry description). It is noted that the nonlinear solution using all fourth-order terms agrees quite well with experimental results in both cases. With the inclusion of the fourth-order terms, exact agreement is obtained with Bushnell's[3] results. Particular improvement in the solution in Fig. 1 is noted in the region two to three inches from the apex where the largest rotations occur. In Fig. 2, the change in stresses between the third and fourth-order theory is less marked. This is due to the fact that for the problem in Fig. 2, the maximum rotation in the cap region is much smaller than that in Fig. 1 (0.018 rad and 0.028 rad, respectively) and hence one would expect a smaller change in stresses for the problem in Fig. 2. For the problem in Fig. 1, the third-order theory overestimates the outside stress at the apex by approximately 13%. Linear theory overestimates the apex stress by over 50%.

Figure 3 presents linear and nonlinear axial displacements for a shallow shell of sandwich construction under localized loading. It is seen that the nonlinear theory with fourth-order terms neglected overestimates the maximum displacement by about 14%. The reason for such a significant

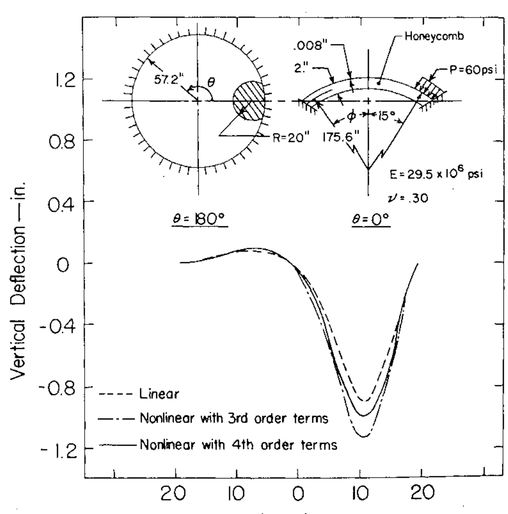

Fig. 3 Deflection along axis of symmetry.

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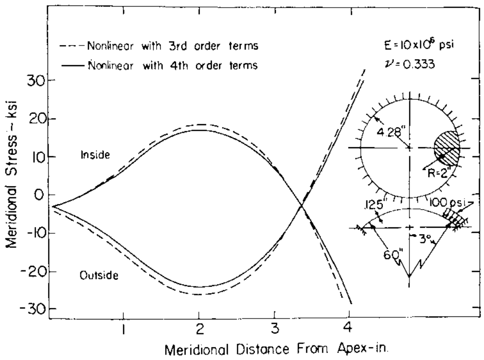

Fig. 4 Meridional stress along arc length.

difference becomes obvious when one compares the relative magnitudes of the strains and rotations. For example, at the point of maximum rotation of the shell, the midsurface strain $\hat{e}_s$ is approximately 0.0037 in./in. while the rotation $\hat{e}_{13}$ is 0.06 rad. Comparing the third-order term $\hat{e}_s\hat{e}_{13}^2$ to the fourth-order term $\hat{e}_{13}^4$, one finds that they contribute almost equally to the strain energy.

Figure 4 presents a comparison of the stresses obtained for a spherical cap under localized loading by the two nonlinear approximations. Again, it is observed that the fourth-order terms are significant, yielding an approximate 8% reduction in stresses when included in the analysis.

Figure 5 presents stress resultants for a shell with negative Gaussian curvature. The omission of the fourth-order terms results in an overestimation of the maximum inside stress by over 17% when compared to the solution obtained with the fourth-order terms. Again, it is noted that the largest deviation between the two approximations occurs in the apex region where the maximum rotations occur.

The importance of the fourth-order terms becomes quite evident in buckling analyses where the prebuckling deformations are large. For the problem in Fig. 5, retention of the fourth-order terms yields a symmetric buckling load of 290 psi whereas neglecting these terms yields a symmetric buckling load of 90 psi.

Based on the results presented here, it is concluded that the fourth-order terms in the strain energy expression previously neglected in Ref. 1 are indeed quite important and hence must be included in the large deflection shell analysis.

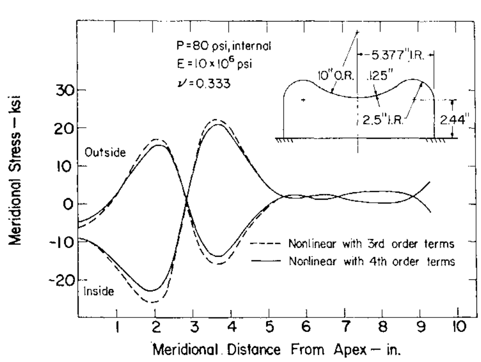

Fig. 5 Meridional stress along arc length.

References

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Time Variations of Generalized Spectral Functions for Density Turbulence

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Introduction

In a previous Note,[1] we illustrated the time variation of the over-all spectrum for a velocity fluctuation turbulence. The energy spectrum varied as k^4 as $k \to 0$. Here we look at the turbulence spectrum for a scalar field, such as the density or temperature fluctuation in a gas. For illustration, the scalar is specified to be electron density fluctuations in an ionized gas imbedded in a neutral background, although we neglect chemical effects and assume isotropic and homogeneous turbulence. For generality, we allow the spectrum to vary as some even power of k as $k \to 0$.

Electron Density Fluctuations

We denote the spectrum of electron density fluctuation by $S(k)$, normalized so that $\int S(k)k^2dk = 2\pi^2$. The energy spectrum is proportional to $k^2S(k)$, and $\langle(\delta n_e)^2\rangle$ is the mean square fluctuation in electron density. The generalized spectrum that we propose to use[2] as a model has the form

$$\langle(\delta n_e)^2\rangle S(k,t) = \langle(\delta n_e)^2\rangle r_0^3(2\pi)^{3/2}n! \frac{(kr_0)^{2n}[k^2r_0^2 + k_0^2r_0^2]^{(-\mu-2-2n)/4}}{2^{-n}(2n+1)!(k_0r_0)^{(-\mu+1)/2}} \frac{K_{\frac{1}{2}\mu+1+n}\{[k^2r_0^2+k_0^2r_0^2]^{1/2}\}}{K_{(\mu-1)/2}(k_0r_0)} \quad (1)$$

where $S(k) \propto k^{-\mu-2}$ in an inertial range where $k_0 \ll k \ll r_0^{-1}$, and $S(k) \propto k^{2n}$ as k goes to zero, where n is a positive integer. We allow time variations in k_0, r_0 and in $\langle(\delta n_e)^2\rangle$, where k_0^{-1} and r_0 now refer to the scale lengths of the density fluctuation. The micro- and integral scale lengths are given by[2]

$$\lambda_{\delta n}^2 = 2(k_0r_0)^{(\mu-1)/2}K_{(\mu-1)/2}(k_0r_0)/[k_0^2(1+2n/3) \times (k_0r_0)^{(\mu-3)/2}K_{(\mu-3)/2}(k_0r_0)] \quad (2)$$

$$\Lambda_{\delta n} = \pi n!(k_0r_0)^{\mu/2}K_{\mu/2}(k_0r_0)/[2^{3/2}\Gamma(n+3/2) \times k_0(k_0r_0)^{(\mu-1)/2}K_{(\mu-1)/2}(k_0r_0)] \quad (3)$$

The Loitsianskii invariant based on this model is[2]

$$I_{\delta n} \equiv \lim_{k\to 0}\left[\frac{\langle(\delta n_e)^2\rangle S(k)}{2\pi^2k^{2n}}\right] = \langle(\delta n_e)^2\rangle\left(\frac{2}{\pi}\right)^{1/2} \times \frac{2^n n!}{(2n+1)!} \frac{(k_0r_0)^{(\mu/2)+1+n}K_{(\mu/2)+1+n}(k_0r_0)}{k_0^{2n+3}(k_0r_0)^{(\mu-1)/2}K_{(\mu-1)/2}(k_0r_0)} \quad (4)$$

The rate of dissipation of density fluctuations[3] $\epsilon_{\delta n}$ has dimensions $T^{-1}L^{-6}$ and is given by $\epsilon_{\delta n} \equiv -(d/dt)\langle(\delta n_e)^2\rangle = 12D_a\langle(\delta n_e)^2\rangle/\lambda_{\delta n}^2$, where D_a is the ambipolar diffusion coefficient. In this discussion, we let ν represent any combination of kinematic viscosity of the background gas and diffusion coefficient of the density, such as $\nu^{1-\alpha}D_a^{\alpha}$, having dimensions L^2/T.

We allow power-time dependences in the two limits of the initial and late decay periods, such that $\epsilon \propto t^{-\gamma}$, $\epsilon_{\delta n} \propto t^{-\xi}$, and $(k_0r_0) \propto (\epsilon t^2/\nu)^x$, where ϵ is the energy dissipation rate per unit mass of the background gas. The results for these

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